Exercise 24

Sketch the graph of a function g for which $g(0) = g(2) = g(4) = 0$, $g'(1) = g'(3) = 0$, $g'(0) = g'(4) = 1$, $g'(2) = -1$, $\lim_{x\to\infty} g(x) = \infty$, and $\lim_{x\to-\infty} g(x) = -\infty$.

Solution

There are eight conditions to be satisfied, so there are eight constants to be determined in the unknown function. The last two limits can be satisfied by using an odd-degree polynomial function with a positive leading coefficient. Let the function be a seventh-degree polynomial for simplicity.

$$g(x) = Ax^7 + Bx^6 + Cx^5 + Dx^4 + Ex^3 + Fx^2 + Gx + H$$

Take the derivative of $g(x)$.

$$g'(x) = 7Ax^6 + 6Bx^5 + 5Cx^4 + 4Dx^3 + 3Ex^2 + 2Fx + G$$

Now apply the given conditions to obtain a system of eight equations for the eight unknowns.

$$\begin{cases} g(0) = A(0)^7 + B(0)^6 + C(0)^5 + D(0)^4 + E(0)^3 + F(0)^2 + G(0) + H = 0 \\ g(2) = A(2)^7 + B(2)^6 + C(2)^5 + D(2)^4 + E(2)^3 + F(2)^2 + G(2) + H = 0 \\ g(4) = A(4)^7 + B(4)^6 + C(4)^5 + D(4)^4 + E(4)^3 + F(4)^2 + G(4) + H = 0 \\ g'(1) = 7A(1)^6 + 6B(1)^5 + 5C(1)^4 + 4D(1)^3 + 3E(1)^2 + 2F(1) + G = 0 \\ g'(3) = 7A(3)^6 + 6B(3)^5 + 5C(3)^4 + 4D(3)^3 + 3E(3)^2 + 2F(3) + G = 0 \\ g'(0) = 7A(0)^6 + 6B(0)^5 + 5C(0)^4 + 4D(0)^3 + 3E(0)^2 + 2F(0) + G = 1 \\ g'(4) = 7A(4)^6 + 6B(4)^5 + 5C(4)^4 + 4D(4)^3 + 3E(4)^2 + 2F(4) + G = 1 \\ g'(2) = 7A(2)^6 + 6B(2)^5 + 5C(2)^4 + 4D(2)^3 + 3E(2)^2 + 2F(2) + G = -1 \end{cases}$$

Solving this system yields

$$\begin{cases} A = \dfrac{1}{480} \\ B = -\dfrac{7}{240} \\ C = \dfrac{61}{480} \\ D = -\dfrac{5}{48} \\ E = -\dfrac{41}{120} \\ F = -\dfrac{1}{60} \\ G = 1 \\ H = 0 \end{cases}.$$

Now that the constants are known, the function is known and can be plotted versus x.

$$g(x) = \frac{1}{480}x^7 - \frac{7}{240}x^6 + \frac{61}{480}x^5 - \frac{5}{48}x^4 - \frac{41}{120}x^3 - \frac{1}{60}x^2 + x$$